ON SUBSPACES CONTAINED IN SUBSETS OF FINITE HOMOGENEOUS SPACES

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Received 22 April 1976
Revised 22 June 1977

Let \( E(1) \subset E(1+1) \subset \cdots \subset E(n) \) be a system of finite sets and \( H(1) \subset H(1+1) \subset \cdots \subset H(n) \) be a system of groups, where \( H(k) \) is a transitive group of automorphisms of \( E(k) \). Denote \( G(k, E(n)) = \{ X \subset E(n): \exists h \in H(n), h(E(k)) = X \} \). We investigate the following problem: given \( n, 1 \leq i \leq k \leq n, 0 < \lambda \leq |E(k)| \) (\(|E(k)|\) is the cardinality of \( E(k) \)), what is the maximal cardinality \( L(n, k, \lambda) \) of a set \( M \subset E(n) \) such that for all \( X \subset E(k), \exists h \in H(n), |X \cap M| < \lambda \)? We shall establish an upper bound for \( L(n, k, \lambda) \) and prove that for some important cases it will coincide with the lower bound \( L(n, k, \lambda) \). We shall consider the three special cases of our problem: linear spaces, Grassmann spaces, Taran's problem. For linear spaces, we obtain the exact formula for the maximal cardinality \( L(n, k, q^k - 1) \) of a subset \( M \subset E^k \), where \( M \) does not contain any \( k \)-subspace of \( E^k \). We shall consider also some applications of this result.

1. Introduction

Let \( E(1) \subset E(1+1) \subset \cdots \subset E(n) \) be a system of finite sets and \( H(1) \subset H(1+1) \subset \cdots \subset H(n) \) be a system of groups, where \( H(k) \) is a transitive group of automorphisms of \( E(k) \). Denote \( G(k, E(n)) = \{ X \subset E(n): \exists h \in H(n), h(E(k)) = X \} \).

We investigate the following problem: given \( n, 1 \leq i \leq k \leq n, 0 < \lambda \leq |E(k)| \) (\(|E(k)|\) is the cardinality of \( E(k) \)), what is the maximal cardinality \( L(n, k, \lambda) \) of a set \( M \subset E(n) \) such that for all \( X \subset E(k), \exists h \in H(n), |X \cap M| < \lambda \)? We shall consider particularly three special cases of this problem: linear spaces, Grassmann spaces, finite graphs.

(i) For linear spaces, we let \( E(n) = E^k \) be a linear \( n \)-space over \( GF(q) \) without 0. Let \( \{ e_i \} (i = 1, \ldots, n) \) be some basis in \( E^k \) and \( E_i \) be a linear span of \( \{ e_i \} \) \( (i = 1, \ldots, k) \). Then we set \( E(k) = E^k \) and \( H(k) = GL(n, k) \) is the group of linear automorphisms such that for \( \forall h \in H(k), h(e_i) = e_i \) \( (i = k+1, \ldots, n) \). For linear spaces, we are interested in the case \( i = 1, \lambda = q^k - 1 \) and denote for this case \( L(n, k, q^k - 1) = L_q(n, k) \).

(ii) Using the same notations as in (i), let \( H(k) = GL(n, k) \) \( (k = 1, \ldots, n) \) and \( E(k) = G(l, E^k) \) be the set of all \( l \)-subspaces in \( E^k \) (Grassmann space). For Grassmann spaces, we set \( \lambda = |G(l, E^k)| \) and denote \( L(n, k, |G(l, E^k)|) = L_q(n, k, l) \).
(iii) Let \( G_\lambda \) be a complete \( n \)-graph with vertices \( e_1, \ldots, e_n \) and \( G_k \) (\( k = l, \ldots, n \)) be a complete subgraph of \( G_\lambda \) with vertices \( e_1, \ldots, e_k \). Then \( E(k) \) is the set of edges of \( G_k \). \( H(k) \) is the group of automorphisms of vertices of \( G_k \) such that for \( \forall h \in H(k) \), \( h(e_i) = e_{i+1} \) (\( i = k+1, \ldots, n \)); \( k = l, \ldots, n \).

For this case, we set \( l = 1 \) and \( L(n, k, \lambda) + 1 = f_k(n, k, \lambda) \). We note that determination of \( f_k(n, k, \lambda) \) is the well-known problem of Turan and values of \( f_k(n, k, \lambda) \) are known only for some special cases [2, 6]. For example, \( f_2(n, 4, 4) \) is an open problem [2].

In Section 3, we shall obtain an upper bound for \( L(n, k, \lambda) \) for the general case of homogeneous spaces which will coincide with lower bounds for \( L(n, k, \lambda) \) for linear spaces, Grassmann spaces with \( l = 1 \) and some cases of Turan's problem. Since the case of linear spaces is the most interesting for us, we establish, in Section 2, the exact value of \( L_k(n, k, \lambda) \), and consider some corollaries from this result. Here we use the direct and shorter proof of the upper bound for \( L_\lambda(n, k) \) due to the referee of this paper.

We note that the main difference between our problem and analogous "Ramsey-type" problems (see, e.g. [3]) is that we are looking for elements from \( G(k, E(n)) \) in a given set \( M \subset E(n) \) and not in one of two sets \( (M \cup E(n) - M) \).

2. Subspaces contained in subsets of linear spaces

In this section, we shall obtain the exact formula for the maximal cardinality \( L_\lambda(n, k) \) of a set \( M \subset E_\lambda^n - 0 \) such that \( M \) does not contain any \( X_\lambda^k - 0 \) where \( X_\lambda^k \) is a \( k \)-subspace of \( E_\lambda^n \).

**Theorem 2.1.** For \( 1 \leq k \leq n \)

\[
L_\lambda(n, k) = q^k - (q^{k+1} - 1)(q - 1)^{-1} - 1. \tag{1}
\]

**Proof.** *Lower bound.* We fix \( X_\lambda^{n-k+1} \) and \( M \subset E_\lambda^n - 0 \) such that for \( \forall X_\lambda^{k+1} \subset X_\lambda^{n-k+1} \), \( |X_\lambda^k \cap M| < q - 1 \). Then

\[
|M| < q^n - (q^{n-k+1} - 1)(q - 1)^{-1} - 1.
\]

Now, let \( X_\lambda^k \) be a subspace of maximal dimension such that \( X_\lambda^k - 0 \subset M \). If \( r \geq k \), then \( \exists X_\lambda^r \) such that \( X_\lambda^r \subset X_\lambda^k \cap X_\lambda^{-k+1} \) and this contradicts the choice of \( M \).

*Upper bound.* We use induction on \( k \). For \( k = 1 \), the result is trivial as each \( X_\lambda^k - 0 \) has to contain an element that is not in \( M \). Suppose, therefore, that

\[
L_\lambda(n, k-1) = q^k - (q^{k+2} - 1)(q - 1)^{-1} - 1.
\]

Suppose \( M \) contains no \( X_\lambda^k - 0 \). If \( M \) contains no \( X_\lambda^{k+1} - 0 \), we are done by induction. Let, therefore \( V - 0 \subset M \), \( V \) is a \( (k-1) \)-dimensional subspace of \( E_\lambda^n \). Write \( E_\lambda^n = V + W \) (direct sum of \( V \) and \( W \)). For each \( 1 \)-dimensional subspace \( T \) of \( W \), choose \( a_T \in T - 0 \). Since \( M \) contains no \( X_\lambda^{k+1} - 0 \), for each \( a_T \), there must be a \( v_T \in V \) and \( \alpha_T \in GF(q) - 0 \) such that \( v_T + \alpha_T a_T \notin M \). If \( v_T - \alpha_T a_T \),
then \( v_T - v_T = \alpha_1 - \alpha_T \) and hence, there are \((q^{n-k+1} - 1)q(q - 1)^{-1}\) distinct elements \(v_T + \alpha_1 - \alpha_T \neq M\) and since \(0 \not\in M\), we have (1).

This proof implies the following result.

**Corollary 2.2.** Let \( M \subset E_q^+ \) and \( M \) contains \( X_q^{k-1} \).

If

\[ |M| > q^n - (q^{n-k+1} - 1)(q - 1)^{-1}, \]

then there exists \( X_q \), such that \( X_q \), \( 0 \leq M \) and \( X_q \), \( 0 \leq X_q \).

If \( M \subset M_1 \subset \cdots \subset M_r \subset E_q^+ \) and \( M \), \( (M_1, \ldots, M_r) \) is an \( r \)-flag of

sets and if \( X_q \subset \cdots \subset X_q \), we say that \( X_q^* \), \( (X_q^*, \ldots, X_q^*) \) is an \( r \)-flag of spaces.

We write \( X_q^* \), \( 0 \leq M \), \( X_q^* \), \( 0 \leq M \), \( (i = 1, \ldots, r) \).

**Corollary 2.3.** If \( M \), \( (M_1, \ldots, M_r) \) is an \( r \)-flag of sets and

\[ |M_i| > q^n - (q^{n-k+1} - 1)(q - 1)^{-1}, \]

then there exists an \( r \)-flag of spaces \( X_q^* \), \( (X_q^*, \ldots, X_q^*) \) such that \( X_q^* \), \( 0 \leq M \).

The proof follows immediately from Corollary 2.2.

Let \( E_q^+ \) be a linear infinite-dimensional space over \( GF(q) \) consisting of all the finite sequences \( E_q^+ = \{a_1, \ldots, a_n, 0, 0, \ldots\} \) : \( a_i \), \( GF(q); i = 1, \ldots, n \), \( n = 1, 2, \ldots \), and \( E_q^+ = \{a_1, \ldots, a_n, 0, 0, \ldots\} \) : \( a_i \), \( GF(q); i = 1, \ldots, n \) be an \( n \)-dimensional subspace of \( E_q^+ \).

**Corollary 2.4.** If \( M_n \subset E_q^+ \) is such that \( 0 \in M_n \) and \( \lim_{n \to \infty} \gamma_n = 1 \), where \( \gamma_n = q^n |M_n \cap E_q| (n = 1, 2, \ldots) \), then there exists an infinite-dimensional space \( E \) such that \( E \subset M_n \).

**Proof.** It is sufficient to prove that there exists a sequence of numbers \( n_0, n_{n_0}, \ldots \) and a sequence of subspaces \( X_q^k \subset E_q^+ \) \( (k = 1, 2, \ldots) \), such that \( X_q^k \subset (M_l \cap E_q) \) and \( X_q^{k+1} \subset X_q^k \). Then we may put \( E = \bigcup_{k \geq n_0} X_q^k \). We construct \( n_0 \) and \( X_q^k \) by induction on \( k \).

Choose \( n_0 \), such that \( \gamma_{n_0} > 1 - (q - 1)^{-1} \).

Then by Theorem 2.1, there exists \( X_q \), \( 0 \leq X_q \). Suppose we have found \( n_0 < n_1 < \cdots < n_k \), and \( X_q^k \subset \cdots \subset X_q \subset X_q \), \( i = 1, \ldots, k \). Choose \( n_{k+1} \) from the condition:

\[ \gamma_{n_k} > 1 - q^{k+1}(q - 1)^{-1} + q^n(q - 1)^{-1}. \]

and by Theorem 2.1, there exists \( X_q^k \), \( 0 \leq X_q^k \). Suppose we have found \( n_0 < n_1 < \cdots < n_k \), and \( X_q^k \subset \cdots \subset X_q \) \( (i = 1, \ldots, k - 1) \). Choose \( n_k + 1 \), such that

\[ \gamma_{n_k} > 1 - q^{k+1}(q - 1)^{-1} + q^n(q - 1)^{-1}. \]

Let us note that no analogue of this corollary for the case of fields \( R \) or \( C \) is known though it is of a great interest in functional analysis; while Ramsey analogues of Theorem 2.1 are well-known for these fields [4, 5].

3. Subspaces contained in subsets of finite homogeneous spaces

In this section, we investigate \( L(n, k, \lambda) \) for the general case of homogeneous spaces (see Section 1).
Lemma 3.1. For any $M \subseteq E(n)$ and $1 \leq t \leq n$,

$$\frac{|M|}{|E(n)|} = \frac{1}{|G(t, E(n))||E(t)|} \sum_{X \subseteq G(t, E(n))} |X \cap M|.$$  

(2)

Proof. In the sum on the right side of (2), each element of $M$ is counted as many times as there are $X \subseteq G(t, E(n))$ which contain this element, i.e., $|E(t)||E(n)|^{-t}|G(t, E(n))|$ times.

We note that left and right sides of (2) are equal to the invariant normalized measure of $M \subseteq E(n)$; and (2) follows from uniqueness of this measure on $E(n)$.

Corollary 3.2. For $1 \leq k \leq n, 1 \leq \lambda \leq |E(k)|$,

$$I(n, k, \lambda) = (\lambda - 1)|E(n)||E(k)|^{-1}.$$  

(3)

The proof follows from Lemma 3.1 with $t = k$.

By $N(m, k, \lambda)$, we denote a number of $X \subseteq G(k, E(n))$ such that for any $M \subseteq E(n)$ with $|M| = m$, we have $|X \cap M| \geq \lambda$ and we denote $\mu(m, k, \lambda) = N(m, k, \lambda)|G(k, E(n))|^{-1}$.

Corollary 3.3. For $1 \leq m \leq |E(n)|$, $1 \leq k \leq n$, $1 \leq \lambda \leq |E(k)|$,

$$\left(\frac{|E(k)|}{|E(n)|}\right)^{m-\lambda} \leq \mu(m, k, \lambda) \leq |E(k)||E(n)|^{-1}m\lambda^{-1}.$$  

(4)

Proof. By definition of $\mu(m, k, \lambda)$ and (2) with $t = k$,

$$\lambda \mu(m, k, \lambda) = |G(k, E(n))|^{-1} \sum_{X \subseteq G(k, E(n))} |X \cap M| = m|E(k)||E(n)|^{-1}$$

$$\leq (1 - \mu(m, k, \lambda)(\lambda - 1) + \lambda \mu(m, k, \lambda)$$  

(5)

and (4) follows from (5).

Theorem 3.4. If for some function $f(n, k, \lambda)$

$$|E(k)||E((k+1)|^{-1}| \cdots |E(n-1)||E(k)|^{-1}f(n, k, \lambda)\cdots |f| \geq \lambda,$$  

then

$$I(n, k, \lambda) \leq f(n, k, \lambda) - 1.$$  

(6)

(7)

($|a|_n$ denotes the least integer $\geq a$).

Proof. Let $M \subseteq E(n)$ and $|M| = f(n, k, \lambda)$. Then by Lemma 3.1 with $t = n - 1$, $\exists X^{n-1} \subseteq G(n-1, E(n))$ such that

$$|E(n-1)||E(n)|^{-1}f(n, k, \lambda) \leq |X^{n-1} \cap M|.$$  

(8)

Next, using (8) and Lemma 3.1 for $n - 1, M_1 = X^{n-1} \cap M, t = n - 2, \exists X^{n-2} \subseteq X^{n-1}$
\((X^{n-2} \in G(n-2, E(n)))\) such that

\[
\|E(n-2)\| |E(n-1)|^{-1} |E(n-1)| |E(n)|^{-1} f(n, k, \lambda) \leq \|E(n-2)\| |E(n-1)|^{-1} |M_i| \leq |X^{n-2} \cap M_i| \leq |X^{n-2} \cap M|.
\]

Continuing this procedure for all \(t = k\), we find that \(\exists X^k \in G(k, E(n))\) such that

\[
\|E(k)\| |E(k+1)|^{-1} \cdots |E(n-1)| |E(n)|^{-1} f(n, k, \lambda) \cdots \leq |X^k \cap M|.
\]

Hence by (6) and definition of \(L(n, k, \lambda)\), we have (7).

We note that the upper bound from Theorem 2.1 immediately follows from Theorem 3.4 in a view of

\[
\left[ \frac{q^k - 1}{q^{k+1} - 1} \right] \cdots \left[ \frac{q^{k-1} - 1}{q^k - 1} \right] = q^k - 1
\]

\((q \geq 2, i = k \Rightarrow n - 1)\).

For the case (ii) of Grassmann spaces \(E(n) = G(l, E_q^r)\) (see Section 1), we have the following bounds for \(L_q(n, k, l)\).

**Corollary 3.5.** For any \(1 \leq l \leq k \leq n\)

\[
|G(l, E_q^n) - |G(l, E_q^{n-k+l})| \leq L_q(n, k, l) \leq |G(l, E_q^n)| - |G(l, E_q^k)| |G(l, E_q^k)|^{-1}
\]

(9)\n
where \(|G(l, E_q^n)| = \prod_{i=1}^{l-1} (q^i - q)\) is the \(q\)-binomial coefficient,

\[
|G(l, E_q^k)| = \prod_{i=0}^{l-1} (q^i - q^i)(q^i - q^{i+1})^{-1}.
\]

(10)

**Proof.** Define \(M = G(l, E_q^n) - G(l, E_q^{n-k+l})\). Then \(\text{dim} (X_q^k \cap E_q^{n-k+l}) \leq l\), for every \(X_q^k \in G(k, E_q^n)\). So there exists \(X_q^k \subseteq E_q^{n-k+l}, X_q^k \subseteq X_q^k\). But then \(X_q^k \subseteq M\) and we have the lower bound from (9). Upper bound follows from Corollary 3.2 with \(E(k) = G(l, E_q^n)\) and \(k = |G(l, E_q^n)|\).

The difference between the lower and upper bounds (9) is not very large. For example, in the case \(k \to \infty, n-k \to \infty\) from (10) follows that for a large \(l\) and \(q\)

\[
\lim_{k \to \infty, n-k \to \infty} \frac{|G(l, E_q^n)| - |G(l, E_q^{n-k+l})|}{|G(l, E_q^n)|} = \prod_{i=0}^{l-1} (1 - q^{-1})^{-1} = \exp((-q^{-1})^{-1}).
\]

(11)

We note also that by the lower bound (9) and the upper bound from Theorem 3.4, we can establish sometimes the exact value for \(L_q(n, k, l)\). For example

\[
L_q(n, k, l) = |G(l, E_q^n)| - |G(l, E_q^{n-k+l})| = (q^k - q^{n-k+l})(q^{-1})^{-1}.
\]

(12)
We consider now the case (iii) when \( E(n) \) is the set of edges of a complete \( n \)-graph (problem of Turan, see Section 1). Turan [6] proved that if \( n = c(k - 1) + r \) \((c \in \{0, 1, \ldots \}, r \in \{0, \ldots, k - 2\})\), then

\[
 f_3(n, k, (\begin{pmatrix} k \end{pmatrix} / 2)) = 0.5(k - 2)(k - 1)^{-1}(n^2 - r^2) + \left(\begin{pmatrix} r \end{pmatrix} / 2\right) + 1.
\]

(13)

We note here that the exact upper bound for \( f_3(n, k, (\begin{pmatrix} k \end{pmatrix} / 2)) \) follows immediately from Theorem 3.4 since

\[
\left(\begin{pmatrix} k \end{pmatrix} / 2\right)^{-1} \cdot \left(\begin{pmatrix} n - 1 \end{pmatrix} / 2\right)^{-1} (\begin{pmatrix} k - 2 \end{pmatrix} / 2(k - 1)) \cdot (n^2 - r^2) + \left(\begin{pmatrix} r \end{pmatrix} / 2\right) + 1 = \left(\begin{pmatrix} k \end{pmatrix} / 2\right).
\]

(14)

Other examples are given by formulas [4]:

\[
f_3(n, k, [0.25k^2 + u]) = [0.25n^2] + u \quad (u \leq [0.25(k + 1)]).
\]

(15)

\[
f_3(n, k, [0.25k^2 + 0.5(k - 1)]) = [0.25n^2] + [0.5(n - 1)] \quad (k > 4)
\]

(16)

\([\text{[a]} \text{ the greatest integer } \leq a]\). The exact upper bounds from (15) and (16) also follow from Theorem 3.4.

We shall give now one more corollary from Theorem 3.4 which may be useful for determination of \( f_3(n, k, \lambda) \).

**Corollary 3.6.** Let \( f(n, k, \lambda) = L(n, k, \lambda) + 1 \). For any \( 1 \leq s \leq k \leq n, \; 0 < \lambda \leq |E(s)| \)

\[
f(n, k, f(k, s, \lambda)) = f(n, s, \lambda)
\]

(17)

and, if for all \( t \in \{k + 1, \ldots, n\} \)

\[
\|E(t - 1) - E(t)\|^{-1} f(t, s, \lambda) = f(t - 1, s, \lambda).
\]

(18)

then

\[
f(n, k, f(k, s, \lambda)) = f(n, s, \lambda).
\]

(19)

**Proof.** Formula (17) follows from the definition of \( f(n, k, \lambda) \). Let \( M \in E(n) \) and \( |M| = f(n, s, \lambda) \). Then as it was shown in the proof of Theorem 3.4, \( \exists X^s \in G(k, E(n)) \) such that

\[
\|E(k) - E(k + 1)\|^{-1} \cdot \cdot \cdot |E(n - 1) - E(n)|^{-1} f(n, s, \lambda) | \subseteq |X^s \cap M|
\]

and in a view of (18)

\[
f(k, s, \lambda) \subseteq |X^s \cap M|.
\]

Hence,

\[
f(n, k, f(k, s, \lambda)) = f(n, s, \lambda)
\]

and by (17) we have (19).
By Corollary 3.6 with \( s = 4, \lambda = 6 \) and (13) we have, for example,

\[
f_r(n, 5, 9) = f_r(n, 5, 13) = f_r(n, 7, 17) = 3^{r-1}(n^2 - r^2) + \binom{r}{2} + 1
\]

(20)

where \( 0 \leq r \leq 2, n \equiv r (\text{mod } 3) \).

4. Sufficient conditions for existence of linear error-correcting codes with given parameters

Let \( \rho(x, y) \) be a metric in \( E_n^x \). A linear \((n, k)\)-code with base \( q \) and distance \( d \) in the metric \( \rho \) is defined as a \( a \) subspace \( X^a_n \) such that \( \min_{a \in X^a_n, b \in E_n^x} \rho(a, b) = d \).

**Theorem 4.1.** For any \( n, k < n \), and any \( Q \subseteq E_n^x(0 \in Q) \), a sufficient condition for the existence of an \((n, k)\)-code \( X^a_n \) with base \( q \), distance \( d \) in the metric \( \rho \), such that \( X^a_n \subseteq Q \), is that

\[
|Q - \{a: \|a\|_w < d\}| > q^n - (q - 1)^{-(q^n - k - 1)}(q - 1)^{-1} (\|a\|_w = \rho(a, 0)).
\]

(21)

**Proof.** A subspace \( X^a_n \) is an \((n, k)\)-code with distance \( d \) in the metric \( \rho \) if \( X^a_n \cap \{a: \|a\|_w < d\} = 0 \). We therefore set \( M = Q - \{a: 0 \in Q, \|a\|_w < d\} \). Then \( |M| = 1 + |Q - \{a: \|a\|_w < d\}| \), and by Theorem 2.1, if (21) holds, there exists an \((n, k)\)-code \( X^a_n \subseteq Q \) with distance \( d \).

Theorem 4.1 yields sufficient conditions for \((n, k)\)-codes \( X^a_n \) satisfying constraints of the type \( X^a_n \subseteq Q \). For example, let \( Q = \{a: \|a\|_w \leq d + \epsilon\} \), where \( \epsilon \) is the Hamming metric [1].

**Corollary 4.2.** For any \( n \) and \( k < n \), a sufficient condition for the existence of an \((n, k)\)-code \( X^a_n \) with base \( q \) and distance \( d \) in the Hamming metric \( \rho \), such that

\[
\max_{a, b \in Q, \|a\|_w \leq \|b\|_w} \rho(a, b) \leq \epsilon,
\]

is that

\[
\sum_{j=0}^{d-1} \binom{n}{j} (q - 1)^j > q^n - (q^n - k - 1)(q - 1)^{-1}.
\]

(22)

**Proof.** Set \( Q = \{a: \|a\|_w \leq d + \epsilon\} \); then

\[
|Q - \{a: \|a\|_w < d\}| = |\{a: d \leq \|a\|_w \leq d + \epsilon\}| = \sum_{j=d}^{d+\epsilon} \binom{n}{j} (q - 1)^j,
\]

and by Theorem 4.1 there exists an \((n, k)\)-code \( X^a_n \subseteq Q \) with distance \( d \), such that for any \( a \in X^a_n \) we have \( d \leq \|a\|_w \leq d + \epsilon \).

Note that in the case \( \epsilon = n - d \) the condition (22) is very close to the well-known Vanshamov-Gilbert bound [1].
Note added in proof

It came to our attention that the upper bound for $I_\infty(n, k)$ also appeared in the paper by M. Deza and F. Hoffman, IEEE Int. Trans. (July 1977) 517–518.

References