THEOREM

1. A code C can detect up to ℓ errors iff

$$d(C) \geq \ell + 1$$

2. A code C can correct up to ℓ errors iff

$$d(C) \geq 2\ell + 1$$

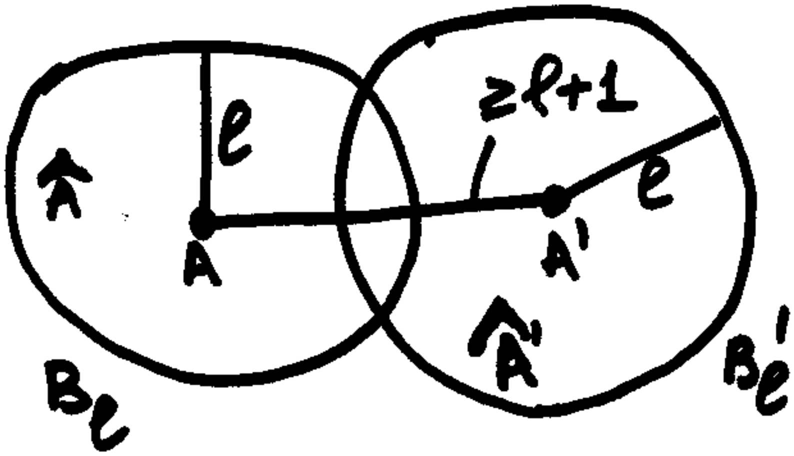
Example If $d(C) = 5$, then C can detect up to 4 errors or C can correct up to 2 error.

Corollary Any code can detect twice more errors that it can correct

GEOMETRICAL INTERPRETATION

I. ERROR DETECTION

ℓ-bit errors



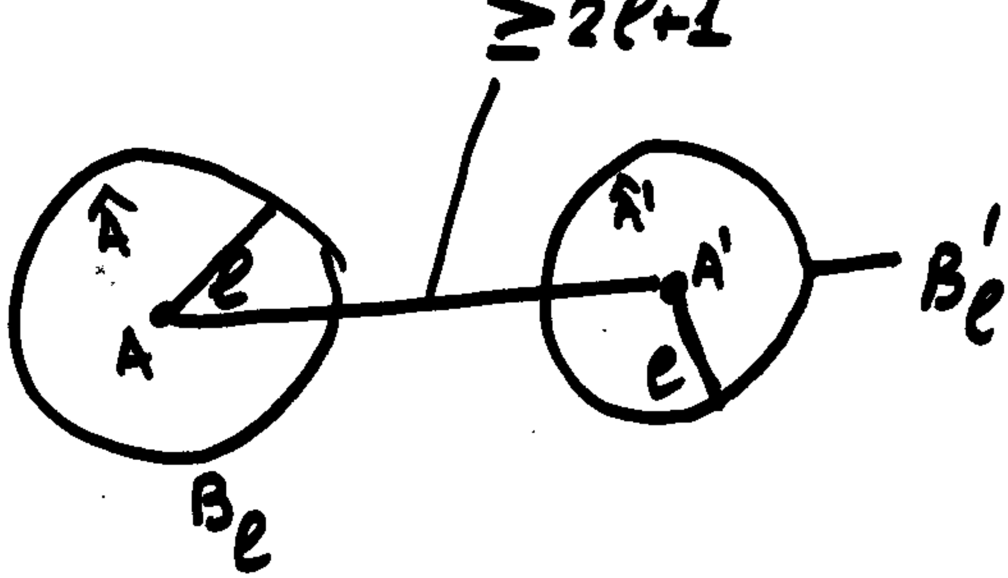
A, A' ERROR FREE OUTPUTS

A, A' $\in C$ = CENTERS OF BALLS B_ℓ and B_ℓ' of radius ℓ

$\hat{A}, \hat{A}'$ - distorted outputs

$d(A, A') \geq \ell + 1.$

I. ERROR CORRECTION



$A, A' \in C$

$\hat{A} \in B_\ell \quad , \quad \widehat{A'} \in B'_\ell$

$d(A, A') \geq 2\ell + 1$

NOTATIONS

(n, M, d) code =

= code of length n,

M codewords,

WITH DISTANCE d.

EXAMPLE $C = \{00\,000,\ 01\,011,\ 10\,110,\ 111\,01\}$ is

$(5,4,3)$ code

ERRORS

FOR BINARY CHANNELS

error free

$$A \xmapsto{\text{error}} \hat{A}$$

distorted

$A, \hat{A} \in Z_2^n$

error : $e = A \oplus \hat{A}$, $e \in Z_2^n$

A	$\hat{A}$	$A \oplus \hat{A}$
0	0	0
0	1	1
1	0	1
1	1	0

$A = 1011$

$\hat{A} = 0110$

$e = 1101$

MULTIPLICITY OF ERRORS :

$$d(A, \hat{A}) = \|A \oplus \hat{A}\| = \|e\|$$

$\hat{A} = A \oplus e$ - additive errors

$\oplus$ - SUM MOD 2 , EXOR

Prob of a single error $- p$
$(\ell=1)$

Prob of any given double error $(\ell=2)$ $-p^2$

Prob of any error with multiplicity ℓ $-p^\ell$

$$p^2 < p \ , \quad p^3 < p^2$$

$(p \leq 1/2)$

FOR q-ary channels

$$A \xmapsto[\text{error}]{} \hat{A} \qquad A, \hat{A} \in Z_q^n$$

$e = \hat{A} \ominus A \pmod q$ error

$\hat{A} = A \oplus e \pmod q \quad e \in Z_q^n$

$\ominus, \oplus$ - are COMPONENTWISE SUBTRACTION and addition of q-ary vectors of length n modulo q

EXAMPLE $q=3$ $n=4$

$A = 1022 \xmapsto[\text{error}]{} \hat{A} = 2012$

$e = 1020$, $\ell = \|e\| = 2$ double error.

multiplicity of error $A \longleftrightarrow \hat{A}$:

$$\ell = \|e\| = \|\hat{A} \ominus A\| = d(\hat{A}, A)$$

- HIGHER THE MULTIPLICITY LESS PROBABLE is THE error.